



Affine symmetric group

Joel Brewster Lewis^{1*}

Abstract

The affine symmetric group is a mathematical structure that describes the symmetries of the number line and the regular triangular tessellation of the plane, as well as related higher dimensional objects. It is an **infinite** extension of the **symmetric group**, which consists of all **permutations** (rearrangements) of a finite set. In addition to its geometric description, the affine symmetric group may be defined as the collection of permutations of the integers $(\dots, -2, -1, 0, 1, 2, \dots)$ that are periodic in a certain sense, or in purely algebraic terms as a **group** with certain **generators and relations**. These different definitions allow for the extension of many important properties of the finite symmetric group to the infinite setting, and are studied as part of the fields of **combinatorics** and **representation theory**.

Definitions

The affine symmetric group, \tilde{S}_n , may be equivalently defined as an abstract group by generators and relations, or in terms of concrete geometric and combinatorial models.

Algebraic definition

In terms of **generators and relations**, \tilde{S}_n is generated by a set

$$s_0, s_1, \dots, s_{n-1}$$

of n elements that satisfy the following relations: when $n \geq 3$,

1. $s_i^2 = 1$ (the generators are **involutions**),
2. $s_i s_j = s_j s_i$ if j is not one of $i - 1, i, i + 1$, and
3. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

In the relations above, indices are taken **modulo** n , so that the third relation includes as a particular case $s_0 s_{n-1} s_0 = s_{n-1} s_0 s_{n-1}$. (The second and third relation are sometimes called the **braid relations**.) When $n = 2$, the affine symmetric group \tilde{S}_2 is the **infinite dihedral group** generated by two elements s_0, s_1 subject only to the relations $s_0^2 = s_1^2 = 1$.^[1]

This definition endows \tilde{S}_n with the structure of a **Coxeter group**, with the s_i as Coxeter generating set. For $n \geq 3$, its **Coxeter–Dynkin diagram** is the n -cycle, while for $n = 2$ it consists of two nodes joined by an edge labeled ∞ .^[2]

Non-technical summary

Flat, straight-edged shapes (like triangles) or 3D ones (like pyramids) have only a finite number of symmetries. In contrast, the affine symmetric group is a way to mathematically describe all the symmetries possible when an infinitely large flat surface is covered by triangular tiles. As with many subjects in mathematics, it can also be thought of in a number of ways: for example, it also describes the symmetries of the infinitely long number line, or the possible arrangements of all integers $(\dots, -2, -1, 0, 1, 2, \dots)$ with certain repetitive patterns. As a result, studying the affine symmetric group extends the study of symmetries of straight-edged shapes or of groups of permutations to the infinite case. It also connects several topics in mathematics that were originally studied for independent reasons, ranging from complex reflection groups to juggling sequences.

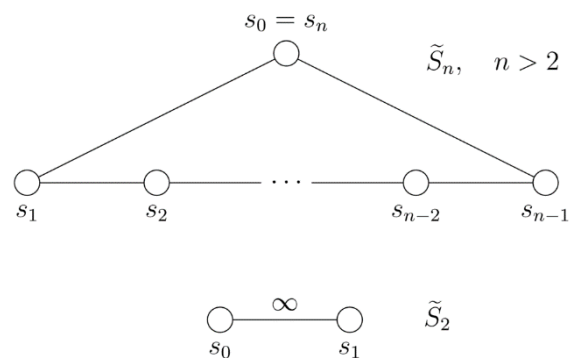


Figure 1 | Dynkin diagrams for the affine symmetric groups on 2 and more than 2 generators

¹ George Washington University

*Author correspondence: jblewis@gwu.edu

ORCID: [0000-0003-0205-8049](https://orcid.org/0000-0003-0205-8049)

Licensed under: [CC-BY](https://creativecommons.org/licenses/by/4.0/)

Received 06-06-2020; accepted 21-04-2021



Geometric definition

In the Euclidean space \mathbb{R}^n with coordinates (x_1, \dots, x_n) , the set V of points that satisfy the equation $x_1 + x_2 + \dots + x_n = 0$ forms a (hyper)plane (an $(n - 1)$ -dimensional subspace). For every pair of distinct elements i and j of $\{1, \dots, n\}$ and every integer k , the set of points in V that satisfy $x_i - x_j = k$ forms a plane in V , and there is a unique reflection of V that fixes this plane. Then the affine symmetric group can be realized geometrically as the collection of all maps from V to itself that arise by composing several of these reflections.^[3]

Inside V , the type A root lattice Λ is the subset of points with integer coordinates, that is, it is the set of all the integer vectors (a_1, \dots, a_n) such that $a_1 + \dots + a_n = 0$. Each of the reflections preserves this lattice, and so the lattice is preserved by the whole group. In fact, one may define \tilde{S}_n to be the group of rigid transformations of V that preserve the lattice Λ .

These reflecting planes divide the space V into congruent simplices, called alcoves.^[4] The situation when $n = 3$ is shown at right; in this case, the root lattice is a triangular lattice, and the reflecting lines divide the plane into equilateral triangular alcoves. (For larger n , the alcoves are not regular simplices.)

To translate between the geometric and algebraic definitions, fix an alcove and consider the n hyperplanes that form its boundary. For example, there is a unique alcove (the *fundamental alcove*) consisting of points (x_1, \dots, x_n) such that $x_1 \geq x_2 \geq \dots \geq x_n \geq x_1 - 1$, which is bounded by the hyperplanes $x_1 - x_2 = 0, x_2 - x_3 = 0, \dots$, and $x_1 - x_n = 1$. (This is illustrated in the case $n = 3$ at right.) For $i = 1, \dots, n - 1$, one may identify the reflection through $x_i - x_{i+1} = 0$ with the Coxeter generator s_{i_i} , and also identify the reflection through $x_1 - x_n = 1$ with the generator $s_0 = s_n$.^[4]

Combinatorial definition

The affine symmetric group may be realized as a group of periodic permutations of the integers. In particular, say that a bijection $u: \mathbb{Z} \rightarrow \mathbb{Z}$ is an affine permutation if $u(x + n) = u(x) + n$ for all integers x and $u(1) + u(2) + \dots + u(n) = 1 + 2 + \dots + n$. (It is a consequence of the first property that the numbers $u(1), \dots, u(n)$ must all be distinct modulo n .) Such a function is uniquely determined by its window notation $[u(1), \dots, u(n)]$, and so affine permutations may also be identified

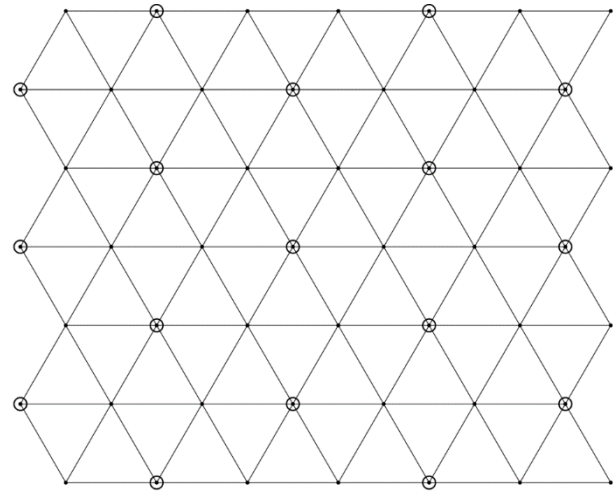


Figure 2 | When $n = 3$, the space V is a two-dimensional plane and the reflections are across lines. The points of the type A root lattice are circled.

with tuples $[u(1), \dots, u(n)]$ of integers that contain one element from each congruence class modulo n and sum to $1 + 2 + \dots + n$.^[5]

To translate between the combinatorial and algebraic definitions, for $i = 1, \dots, n - 1$ one may identify the Coxeter generator s_i with the affine permutation that has window notation $[1, 2, \dots, i - 1, i + 1, i, i + 2, \dots, n]$, and also identify the generator $s_0 = s_n$ with the affine permutation $[0, 2, 3, \dots, n - 2, n - 1, n + 1]$. More generally, every reflection (that is, a conjugate of

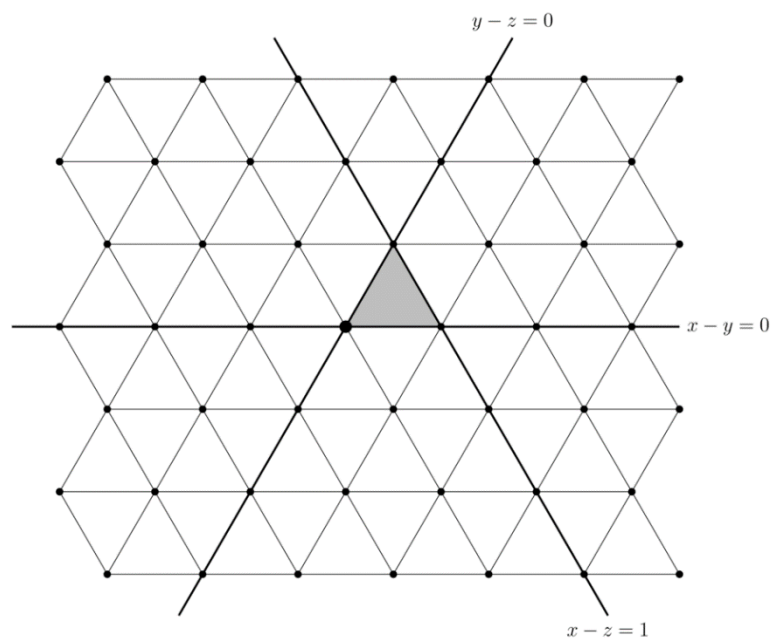


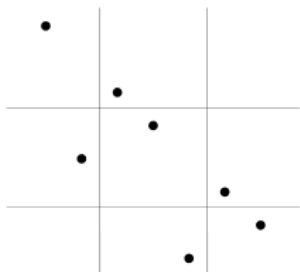
Figure 3 | Reflections and alcoves for the affine symmetric group. The fundamental alcove is shaded.



one of the Coxeter generators) can be described uniquely as follows: for distinct integers i, j in $\{1, \dots, n\}$ and arbitrary integer k , it maps i to $j - kn$, maps j to $i + kn$, and fixes all inputs not congruent to i or j modulo n .^[6] (In terms of the geometric definition, this corresponds to the reflection across the plane $x_i - x_j = k$. The correspondence between the geometric and combinatorial representations for other elements is discussed below.)

Representation as matrices

One may represent affine permutations as infinite periodic **permutation matrices**.^[7] If $u: \mathbb{Z} \rightarrow \mathbb{Z}$ is an affine permutation, one places the entry 1 at position $(i, u(i))$ in the infinite grid $\mathbb{Z} \times \mathbb{Z}$ for each integer i , and all other entries are equal to 0. Since u is a bijection, the resulting matrix contains exactly one 1 in every row and column. The periodicity condition on the map u ensures that the entry at position (a, b) is equal to the entry at position $(a + n, b + n)$ for every pair of integers (a, b) . For example, a portion of matrix for the affine permutation $[2, 0, 4] \in \tilde{S}_3$ is shown below, with the conventions that 1s are replaced by \bullet , 0s are omitted, rows numbers increase from top to bottom, column numbers increase from left to right, and the boundary of the box consisting of rows and columns 1, 2, 3 is drawn:



Relationship to the finite symmetric group

Relationship to the finite symmetric group

The affine symmetric group \tilde{S}_n contains the finite symmetric group S_n as both a **subgroup** and a **quotient**.

As a subgroup

There is a canonical way to choose a subgroup of \tilde{S}_n that is isomorphic to the finite symmetric group S_n . In terms of the algebraic definition, this is the subgroup of \tilde{S}_n generated by s_1, \dots, s_{n-1} (excluding the simple reflection $s_0 = s_n$). Geometrically, this corresponds to the subgroup of transformations that fix the origin, while

combinatorially it corresponds to the window notations for which $\{u(1), \dots, u(n)\} = \{1, 2, \dots, n\}$ (that is, in which the window notation is the **one-line notation** of a finite permutation).^{[8][3]}

If $u = [u(1), u(2), \dots, u(n)]$ is the window notation of an element of this standard copy of $S_n \subset \tilde{S}_n$, its action on the hyperplane V in \mathbb{R}^n is given by permutation of coordinates: $(x_1, x_2, \dots, x_n) \cdot u = (x_{u(1)}, x_{u(2)}, \dots, x_{u(n)})$. (In this article, the geometric action of permutations and affine permutations is on the right; thus, if u and v are two affine permutations, the action of uv on a point is given by first applying u , then applying v .)

There are also many nonstandard copies of S_n contained in \tilde{S}_n . A geometric construction is to pick any point a in Λ (that is, an integer vector whose coordinates sum to 0); the subgroup $(\tilde{S}_n)_a$ of \tilde{S}_n of isometries that fix a is isomorphic to S_n . The analogous combinatorial construction is to choose any subset A of \mathbb{Z} that contains one element from each conjugacy class modulo n and whose elements sum to $1 + 2 + \dots + n$; the subgroup $(\tilde{S}_n)_A$ of \tilde{S}_n of affine permutations that stabilize A is isomorphic to S_n .

As a quotient

There is a simple map (technically, a **surjective group homomorphism**) π from \tilde{S}_n onto the finite symmetric group S_n . In terms of the combinatorial definition, it is to reduce the window entries modulo n to elements of $\{1, 2, \dots, n\}$, leaving the one-line notation of a permutation. The image $\pi(u)$ of an affine permutation u is called the **underlying permutation** of u .

The map π sends the Coxeter generator $s_0 = [0, 2, 3, 4, \dots, n - 2, n - 1, n + 1]$ to the permutation whose one-line notation and **cycle notation** are $[n, 2, 3, 4, \dots, n - 2, n - 1, 1]$ and $(1\ n)$, respectively. In terms of the Coxeter generators of S_n , this can be written as $\pi(s_0) = s_1 s_2 \dots s_{n-2} s_{n-1} s_{n-2} \dots s_2 s_1$.

The **kernel** π is the set of affine permutations whose underlying permutation is the **identity**. The window notations of such affine permutations are of the form $[1 - a_1 \cdot n, 2 - a_2 \cdot n, \dots, n - a_n \cdot n]$, where (a_1, a_2, \dots, a_n) is an integer vector such that $a_1 + a_2 + \dots + a_n = 0$, that is, where $(a_1, \dots, a_n) \in \Lambda$. Geometrically, this kernel consists of the **translations**, that is, the isometries that shift the entire space V without rotating or reflecting it. In an **abuse of notation**, the symbol Λ is used in this article for all three of these sets (integer vectors in V , affine permutations with underlying permutation the identity, and translations); in all three settings, the natural group operation turns Λ into an **abelian**

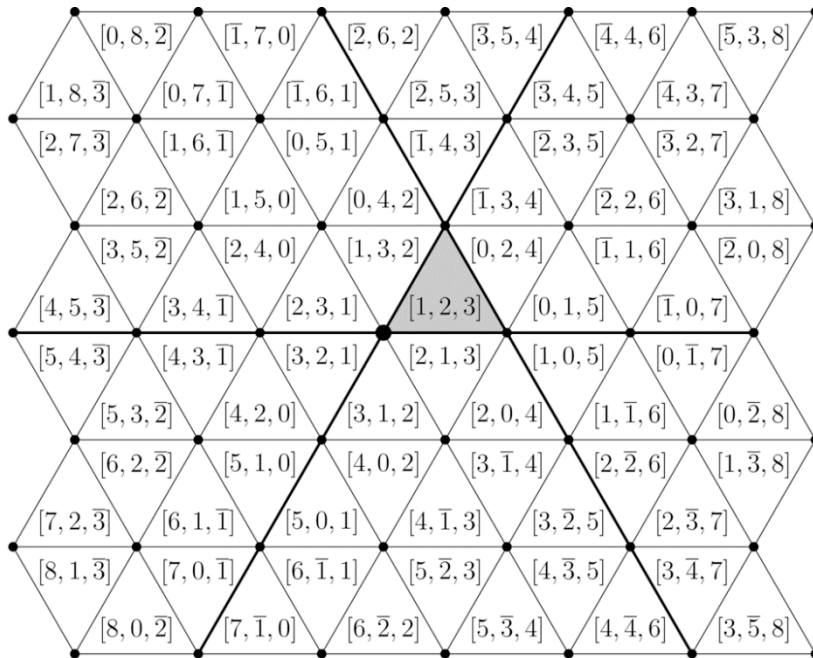


Figure 4 | Alcoves for \tilde{S}_3 labeled by affine permutations. An alcove A is labeled by the window notation for a permutation u if u sends the fundamental alcove (shaded) to A . Negative numbers are denoted by over-bars.

group, generated freely by the $n - 1$ vectors $\{(1, -1, 0, \dots, 0), (0, 1, -1, \dots, 0), \dots, (0, \dots, 0, 1, -1)\}$.

under the action of g . This identification for \tilde{S}_3 is illustrated in Figure 4.

Connection between the geometric and combinatorial definitions

The subgroup Λ is a normal subgroup of \tilde{S}_n , and one has an isomorphism

$$\tilde{S}_n \cong S_n \ltimes \Lambda$$

between \tilde{S}_n and the semidirect product of the finite symmetric group S_n with Λ , where the action of S_n on Λ is by permutation of coordinates. Consequently, identifying the finite symmetric group S_n as its standard copy in \tilde{S}_n , one has that every element u of \tilde{S}_n may be realized uniquely as a product $u = r \cdot t$ where $r \in S_n$ is a finite permutation and $t \in \Lambda$.

This point of view allows for a direct translation between the combinatorial and geometric definitions of \tilde{S}_n : if one writes $[u(1), \dots, u(n)] = [r_1 - a_1 \cdot n, \dots, r_n - a_n \cdot n]$ where $r = [r_1, \dots, r_n] = \pi(u)$ and $(a_1, a_2, \dots, a_n) \in \Lambda$ then the affine permutation u corresponds to the rigid motion of V defined by

$$(x_1, \dots, x_n) \cdot u = (x_{r(1)} + a_1, \dots, x_{r(n)} + a_n).$$

Furthermore, as with every affine Coxeter group, the affine symmetric group acts transitively and freely on the set of alcoves. Hence, by making an arbitrary choice of alcove A_0 , one may place the group in one-to-one correspondence with the alcoves: the identity element corresponds to A_0 , and every other group element g corresponds to the alcove $A = A_0 \cdot g$ that is the image of A_0

Example: $n = 2$

Algebraically, \tilde{S}_2 is the infinite dihedral group, generated by two generators s_0, s_1 subject to the relations $s_0^2 = s_1^2 = 1$. Every other element of the group can be written as an alternating product of copies of s_0 and s_1 .

Combinatorially, the affine permutation s_1 has window notation $[2, 1]$, corresponding to the bijection $2k \mapsto$

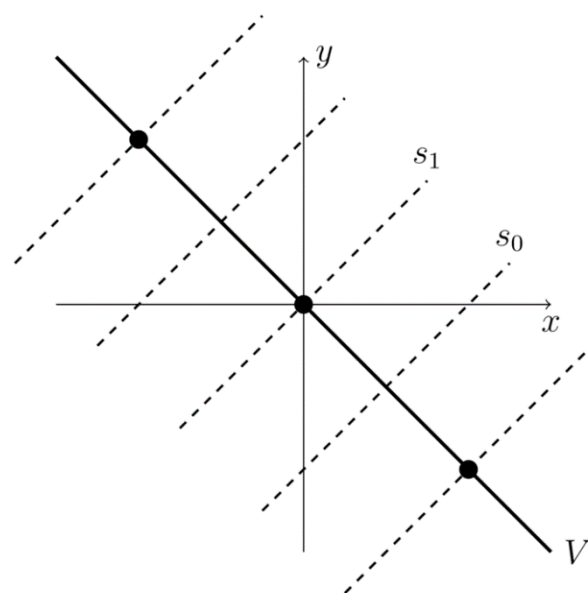


Figure 5 | The affine symmetric group \tilde{S}_2 acts on the line V in the Euclidean plane. The reflections are through the dashed lines. The vectors of the root lattice Λ are marked.



$2k - 1, 2k - 1 \mapsto 2k$ for every integer k . The affine permutation s_0 has window notation $[0, 3]$, corresponding to the bijection $2k \mapsto 2k + 1, 2k + 1 \mapsto 2k$ for every integer k . Other elements have the following window notations:

- $\overbrace{s_0 s_1 \cdots s_0 s_1}^{2k \text{ factors}} = [1 + 2k, 2 - 2k],$
- $\overbrace{s_1 s_0 \cdots s_1 s_0}^{2k \text{ factors}} = [1 - 2k, 2 + 2k],$
- $\overbrace{s_0 s_1 \cdots s_0}^{2k+1 \text{ factors}} = [2 + 2k, 1 - 2k],$
- $\overbrace{s_1 s_0 \cdots s_1}^{2k+1 \text{ factors}} = [2 - 2(k + 1), 1 + 2(k + 1)].$

Geometrically, the space V is the line with equation $x + y = 0$ in the Euclidean plane \mathbb{R}^2 . The root lattice inside V consists of those pairs $(a, -a)$ for integral a . The Coxeter generator s_1 acts on V by reflection across the line $x = y$ (that is, across the origin); the generator s_0 acts on V by reflection across the line $x = y + 1$ (that is, across the point $(\frac{1}{2}, -\frac{1}{2})$). It is natural to identify the line V with the real line \mathbb{R}^1 , by sending the point $(x, -x)$ to the real number $2x$. With this identification, the root lattice consists of the even integers; the fundamental alcove is the interval $[0, 1]$; the element $(s_1 s_0)^k$ acts by translation by k for any integer k ; and the reflection $s_1 (s_0 s_1)^k$ reflects across the point $-k$ for any integer k .

Permutation statistics and permutation patterns

Many permutation statistics and other features of the combinatorics of finite permutations can be extended to the affine case.

Descents, length, and inversions

The length $\ell(g)$ of an element g of a Coxeter group G is the smallest number k such that g can be written as a product $g = s_{i_1} \cdots s_{i_k}$ of k Coxeter generators of G .^[9]

Geometrically, the length of an element g in \tilde{S}_n is the number of reflecting hyperplanes that separate A_0 and $A_0 \cdot g$, where A_0 is the fundamental alcove (the simplex bounded by the reflecting hyperplanes of the Coxeter generators s_0, s_1, \dots, s_{n-1}). (In fact, the same is true for any affine Coxeter group.)^[10]

Combinatorially, the length of an affine permutation is encoded in terms of an appropriate notion of **inversions**. In particular, one has for an affine permutation u that^[11]

$$\ell(u) = \#\{(i, j) : i \in \{1, \dots, n\}, i < j, \text{ and } u(i) > u(j)\}.$$

Alternatively, it is the number of equivalence classes of pairs $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ such that $i < j$ and $u(i) > u(j)$ under the equivalence relation $(i, j) \equiv (i', j')$ if $(i - i', j - j') = (kn, kn)$ for some integer k .

The generating function for length in \tilde{S}_n is^{[12][13]}

$$\sum_{g \in \tilde{S}_n} q^{\ell(g)} = \frac{1 - q^n}{(1 - q)^n}.$$

Similarly, one may define an affine analogue of **descents** in permutations: say that an affine permutation u has a descent in position i if $u(i) > u(i + 1)$. (By periodicity, u has a descent in position i if and only if it has a descent in position $i + kn$ for all integers k .)^[14]

Algebraically, the descents corresponds to the *right descents* in the sense of Coxeter groups; that is, i is a descent of u if and only if $\ell(u \cdot s_i) < \ell(u)$.^[14] The left descents (that is, those indices i such that $\ell(s_i \cdot u) < \ell(u)$) are the descents of the inverse affine permutation u^{-1} ; equivalently, they are the values i such that i occurs before $i - 1$ in the sequence $\dots, u(-2), u(-1), u(0), u(1), u(2), \dots$.

Geometrically, i is a descent of u if and only if the fixed hyperplane of s_i separates the alcoves A_0 and $A_0 \cdot u$.

Because there are only finitely many possibilities for the number of descents of an affine permutation, but infinitely many affine permutations, it is not possible to naively form a generating function for affine permutations by number of descents (an affine analogue of **Eulerian polynomials**).^[15] One possible resolution is to consider affine descents (equivalently, cyclic descents) in the finite symmetric group S_n .^[16] Another is to consider simultaneously the length and number of descents of an affine permutation. The generating function for these statistics over \tilde{S}_n simultaneously for all n is

$$\sum_{n \geq 1} \frac{x^n}{1 - q^n} \sum_{w \in \tilde{S}_n} t^{\text{des}(w)} q^{\ell(w)} = \left[\frac{x \cdot \frac{\partial}{\partial x} \log(\exp(x; q))}{1 - t \exp(x; q)} \right]_{x \mapsto x \frac{1-t}{1-q}}$$

where $\text{des}(w)$ is the number of descents of the affine permutation W and $\exp(x; q) = \sum_{n \geq 0} \frac{x^n (1-q)^n}{(1-q)(1-q^2) \cdots (1-q^n)}$ is the **q -exponential function**.^[17]

Cycle type and reflection length

Any bijection $u: \mathbb{Z} \rightarrow \mathbb{Z}$ partitions the integers into a (possibly infinite) list of (possibly infinite) cycles: for each integer i , the cycle containing i is the sequence $(\dots, u^{-2}(i), u^{-1}(i), i, u(i), u^2(i), \dots)$ where exponentiation represents functional composition. For example, the affine permutation in \tilde{S}_5 with window notation $[6, 3, 2, 0, 4]$ contains the two infinite cycles



$(\dots, -9, -4, 1, 6, 11, \dots)$ and $(\dots, 10, 9, 5, 4, 0, -1, -5 \dots)$ as well as infinitely many finite cycles $(5k + 2, 5k + 3)$ for each $k \in \mathbb{Z}$. Cycles of an affine permutation correspond to cycles of the underlying permutation in an obvious way: in the example above, with underlying permutation $[1, 3, 2, 5, 4] = (1)(23)(45)$, the first infinite cycle corresponds to the cycle (1), the second corresponds to the cycle (45), and the finite cycles all correspond to the cycle (23).

For an affine permutation u , the following conditions are equivalent: all cycles of u are finite, u has finite order, and the geometric action of u on the space V has at least one fixed point.^[18]

The reflection length $\ell_R(u)$ of an element u of \tilde{S}_n is the smallest number k such that there exist reflections r_1, \dots, r_k such that $u = r_1 \cdots r_k$. (In the symmetric group, reflections are transpositions, and the reflection length of a permutation u is $n - c(u)$, where $c(u)$ is the number of cycles of u .^[19]) In (Lewis et al. 2019), the following formula was proved for the reflection length of an affine permutation u : for each cycle of u , define the weight to be the integer k such that consecutive entries congruent modulo n differ by exactly kn . (For example, in the permutation $[6, 3, 2, 0, 4]$ above, the first infinite cycle has weight 1 and the second infinite cycle has weight -1 ; all finite cycles have weight 0.) Form a tuple of cycle weights of u (counting translates of the same cycle by multiples of n only once), and define the nullity $\nu(u)$ to be the size of the smallest set partition of this tuple so that each part sums to 0. (In the example above, the tuple is $(1, -1, 0)$ and the nullity is 2, since one can take the partition $(1, -1), (0)$.) Then the reflection length of u is

$$\ell_R(u) = n - 2\nu(u) + c(\pi(u)),$$

where $\pi(u)$ is the underlying permutation of u .^[20]

For every affine permutation u , there is a choice of subgroup W of \tilde{S}_n such that $W \cong S_n$, $\tilde{S}_n = W \ltimes A$, and for the standard form $u = w \cdot t$ implied by this semidirect product, one has $\ell_R(u) = \ell_R(w) + \ell_R(t)$.

Fully commutative elements and pattern avoidance

A reduced word for an element g of a Coxeter group is a tuple $(s_{i_1}, \dots, s_{i_{\ell(g)}})$ of Coxeter generators of minimum possible length such that $g = s_{i_1} \cdots s_{i_{\ell(g)}}$.^[9] The element g is called fully commutative if one can transform any reduced word into any other by sequentially swapping pairs of factors that commute.^[22] For example, in the finite symmetric group S_4 , the element $2143 = (12)(34)$ is fully commutative, since its two reduced

words (s_1, s_3) and (s_3, s_1) can be connected by swapping commuting factors, but $3214 = (13)(2)(4)$ is not fully commutative because there is no way to reach the reduced word (s_2, s_1, s_2) starting from the reduced word (s_1, s_2, s_1) by commutations.

Billey, Jockusch & Stanley (1993) proved that in the finite symmetric group S_n , a permutation is fully commutative if and only if it avoids the permutation pattern 321, that is, if and only if its one-line notation contains no three-term decreasing subsequence. In (Green 2002), this result was extended to affine permutations: an affine permutation u is fully commutative if and only if there do not exist integers $i < j < k$ such that $u(i) > u(j) > u(k)$.^[3]

It has also been shown that the number of affine permutations avoiding a single pattern p is finite if and only if p avoids the pattern 321,^[24] so in particular there are infinitely many fully commutative affine permutations. These were enumerated by length in (Hanusa & Jones 2010).

Parabolic subgroups and other structures

The parabolic subgroups of \tilde{S}_n and their coset representatives offer a rich combinatorial structure. Other aspects of the affine symmetric group, such as its Bruhat order and representation theory, may also be understood via combinatorial models.

Parabolic subgroups, coset representatives

A standard parabolic subgroup of a Coxeter group is a subgroup generated by a subset of its Coxeter generating set. The maximal parabolic subgroups are those that come from omitting a single Coxeter generator. In \tilde{S}_n , all maximal parabolic subgroups are isomorphic to the finite symmetric group S_n . The subgroup generated by the subset $\{s_0, \dots, s_{n-1}\} \setminus \{s_i\}$ consists of those affine permutations that stabilize the interval $[i + 1, i + n]$, that is, that map every element of this interval to another element of the interval.^[14]

The non-maximal parabolic subgroups of \tilde{S}_n are all isomorphic to parabolic subgroups of S_n , that is, to a Young subgroup $S_{a_1} \times \cdots \times S_{a_k}$ for some positive integers a_1, \dots, a_k with sum n .

For a fixed element i of $\{0, \dots, n - 1\}$, let $J = \{s_0, \dots, s_{n-1}\} \setminus \{s_i\}$ be the maximal proper subset of Coxeter generators omitting s_i , and let $(\tilde{S}_n)_J$ denote the parabolic subgroup generated by J . Every coset $g \cdot$

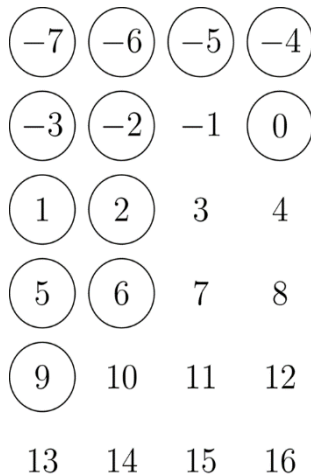


Figure 6 | Abacus diagram of the affine permutation $[-5, 0, 6, 9]$.

$(\tilde{S}_n)_J$ has a unique element of minimum length. The collection of such representatives, denoted $(\tilde{S}_n)^J$, consists of the following affine permutations:^[14]

$$(\tilde{S}_n)^J = \{u \in \tilde{S}_n: u(i - n + 1) < u(i - n + 2) < \dots < u(i - 1) < u(i)\}.$$

In the particular case that $J = \{s_1, \dots, s_{n-1}\}$, so that $(\tilde{S}_n)_J \cong S_n$ is the standard copy of S_n inside \tilde{S}_n , the elements of $(\tilde{S}_n)^J \cong \tilde{S}_n/S_n$ may naturally be represented by *abacus diagrams*: the integers are arranged in an infinite strip of width n , increasing sequentially along rows and then from top to bottom; integers are circled if they lie directly above one of the window entries of the minimal coset representative. For example, the minimal coset representative $u = [-5, 0, 6, 9]$ is represented by the abacus diagram in figure 6. To compute the length of the representative from the abacus diagram, one adds up the number of uncircled numbers that are smaller than the last circled entry in each column. (In the example shown, this gives $5 + 3 + 0 + 1 = 9$.)^[25]

Other combinatorial models of minimum-length coset representatives for \tilde{S}_n/S_n can be given in terms of *core partitions* (integer partitions in which no hook length is divisible by n) or bounded partitions (integer partitions in which no part is larger than $n - 1$). Under these correspondences, it can be shown that the *weak Bruhat order* on \tilde{S}_n/S_n is isomorphic to a certain subposet of *Young's lattice*.^{[26][27]}

Bruhat order

The *Bruhat order* on \tilde{S}_n has the following combinatorial realization. If u is an affine permutation and i and j are integers, define $u[i, j]$ to be the number of integers a such that $a \leq i$ and $u(a) \geq j$. (For example, with $u = [2, 0, 4] \in \tilde{S}_3$, one has $u[3, 1] = 3$: the three relevant

values are $a = 0, 1, 3$, which are respectively mapped by u to 1, 2, and 4.) Then for two affine permutations u, v , one has that $u \leq v$ in Bruhat order if and only if $u[i, j] \leq v[i, j]$ for all integers i, j .^[28]

Representation theory and an affine Robinson–Schensted correspondence

In the finite symmetric group, the *Robinson–Schensted correspondence* gives a bijection between the group and pairs (P, Q) of *standard Young tableaux* of the same shape. This bijection plays a central role in the combinatorics and the *representation theory of the symmetric group*. For example, in the language of *Kazhdan–Lusztig theory*, two permutations lie in the same left cell if and only if their images under Robinson–Schensted have the same tableau Q , and in the same right cell if and only if their images have the same tableau P . In (Shi 1986), J.-Y. Shi showed that left cells for \tilde{S}_n are indexed instead by *tabloids*,^[b] and in (Shi 1991) he gave an algorithm to compute the tabloid analogous to the tableau P for an affine permutation. In (Chmutov, Pylyavskyy & Yudovina 2018), the authors extended Shi's work to give a bijective map between \tilde{S}_n and triples (P, Q, ρ) consisting of two tabloids of the same shape and an integer vector whose entries satisfy certain inequalities. Their procedure uses the matrix representation of affine permutations and generalizes the *shadow construction* of Viennot (1977).

Inverse realizations

In some situations, one may wish to consider the action of the affine symmetric group on \mathbb{Z} or on alcoves that is inverse to the one given above.^[c] We describe these alternate realizations now.

In the combinatorial action of \tilde{S}_n on \mathbb{Z} , the generator s_i acts by switching the *values* i and $i + 1$. In the inverse action, it instead switches the entries in *positions* i and $i + 1$. Similarly, the action of a general reflection will be to switch the entries at i and $i + kn$ for each k , fixing all inputs at positions not congruent to i or j modulo n .^[29] (In the finite symmetric group S_n , the analogous distinction is between the *active* and *passive* forms of a permutation.^[30])

In the geometric action of \tilde{S}_n , the generator s_i acts on an alcove A by reflecting it across one of the bounding planes of the fundamental alcove A_0 . In the inverse action, it instead reflects A across one of *its own* bounding planes. From this perspective, a reduced word corresponds to an *alcove walk* on the tessellated space V .^[31]

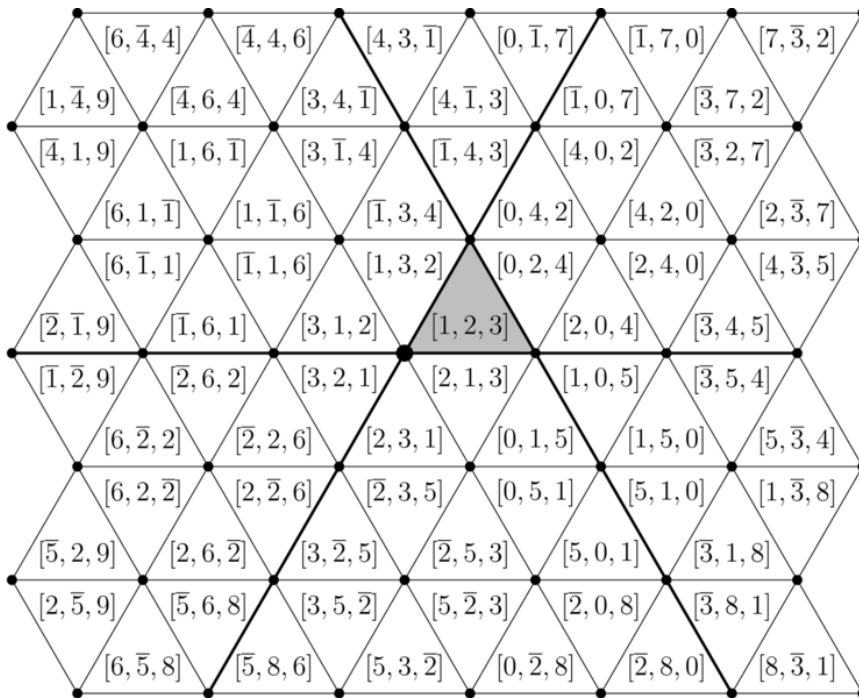


Figure 7 | Alcoves for \tilde{S}_3 labeled by affine permutations, inverse to the labeling above.

Relationship to other mathematical objects

The affine symmetric group is closely related to a variety of other mathematical objects.

Juggling patterns

In (Ehrenborg & Readdy 1996), a correspondence is given between affine permutations and juggling patterns encoded in a version of siteswap notation.^[32] Here, a juggling pattern of period n is a sequence (a_1, \dots, a_n) of nonnegative integers (with certain restrictions) that captures the behavior of balls thrown by a juggler, where the number a_i indicates the length of time the i th throw spends in the air (equivalently, the height of the throw).^[d] The number b of balls in the pattern is the average $b = \frac{a_1 + \dots + a_n}{n}$.^[34] The Ehrenborg–Readdy correspondence associates to each juggling pattern $\mathbf{a} = (a_1, \dots, a_n)$ of period n the function $w_{\mathbf{a}}: \mathbb{Z} \rightarrow \mathbb{Z}$ defined by

$$w_{\mathbf{a}}(i) = i + a_i - b,$$

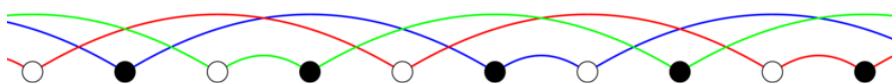


Figure 8 | The juggling pattern 441 visualized as an arc diagram: the height of each throw corresponds to the length of an arc; the two colors of nodes are the left and right hands of the juggler. This pattern has four crossings, which repeat periodically.

where indices of the sequence \mathbf{a} are taken modulo n . Then $w_{\mathbf{a}}$ is an affine permutation in \tilde{S}_n , and moreover every affine permutation arises from a juggling pattern in this way.^[32] Under this bijection, the length of the affine permutation is encoded by a natural statistic in the juggling pattern: one has

$$\ell(w_{\mathbf{a}}) = (b - 1)n - \text{cross}(\mathbf{a}),$$

where $\text{cross}(\mathbf{a})$ is the number of crossings (up to periodicity) in the arc diagram of \mathbf{a} . This allows an elementary proof of the generating function for affine permutations by length.^[35]

For example, the juggling pattern 441 (Figures 8 & 9) has $n = 3$ and $b = \frac{4+4+1}{3} = 3$. Therefore, it corresponds to the affine permutation $w_{441} = [1 + 4 - 3, 2 + 4 - 3, 3 + 1 - 3] = [2, 3, 1]$. The juggling pattern has four crossings, and the affine permutation has length $\ell(w_{441}) = (3 - 1) \cdot 3 - 4 = 2$.

Similar techniques can be used to derive the generating function for minimal coset representatives of \tilde{S}_n/S_n by length.^[36]

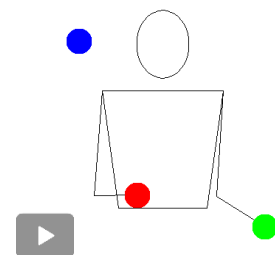


Figure 9 | The juggling pattern 441. For video, follow this link: https://commons.wikimedia.org/wiki/File:Juggling_441.gif Nummer9, CC BY SA 3.0



Complex reflection groups

In a finite-dimensional real **inner product space**, a *reflection* is a **linear transformation** that fixes a linear hyperplane pointwise and negates the vector orthogonal to the plane. This notion may be extended to vector spaces over other **fields**. In particular, in a complex inner product space, a *reflection* is a **unitary transformation** T of finite order that fixes a hyperplane.^[6] This implies that the vectors orthogonal to the hyperplane are eigenvectors of T , and the associated eigenvalue is a complex **root of unity**. A **complex reflection group** is a finite group of linear transformations on a complex vector space generated by reflections.

The complex reflection groups were fully classified by **Shephard & Todd (1954)**: each complex reflection group is isomorphic to a product of irreducible complex reflection groups, and every irreducible either belongs to an infinite family $G(m, p, n)$ (where m , p , and n are positive integers such that p divides m) or is one of 34 other (so-called "exceptional") examples. The group $G(m, 1, n)$ is the **generalized symmetric group**: algebraically, it is the **wreath product** $(\mathbb{Z}/m\mathbb{Z}) \wr S_n$ of the **cyclic group** $\mathbb{Z}/m\mathbb{Z}$ with the symmetric group S_n . Concretely, the elements of the group may be represented by **monomial matrices** (matrices having one nonzero entry in every row and column) whose nonzero entries are all m th roots of unity. The groups $G(m, p, n)$ are subgroups of $G(m, 1, n)$, and in particular the group $G(m, m, n)$ consists of those matrices in which the product of the nonzero entries is equal to 1.

In **(Shi 2002)**, Shi showed that the affine symmetric group is a *generic cover* of the family $\{G(m, m, n) : m \geq 1\}$, in the following sense: for every positive integer m , there is a surjection π_m from \tilde{S}_n to $G(m, m, n)$, and these maps are compatible with the natural surjections $G(m, m, n) \rightarrow G(p, p, n)$ when $p \mid m$ that come from raising each entry to the m/p th power. Moreover, these projections respect the reflection group structure, in that the image of every reflection in \tilde{S}_n under π_m is a reflection in $G(m, m, n)$; and similarly when $m > 1$ the image of the standard **Coxeter element** $s_0 \cdot s_1 \cdots s_{n-1}$ in \tilde{S}_n is a Coxeter element in $G(m, m, n)$.^[37]

Affine Lie algebras

Each affine Coxeter group is associated to an **affine Lie algebra**, a certain **infinite-dimensional non-associative algebra** with unusually nice representation-theoretic properties. In this association, the Coxeter group arises as a group of symmetries of the root space of the Lie algebra (the dual of the Cartan subalgebra).^[38] In the classification of affine Lie algebras, the one associated

to \tilde{S}_n is of (untwisted) type $A_{n-1}^{(1)}$, with **Cartan matrix** $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ for $n = 2$ and

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

(a **circulant matrix**) for $n > 2$.^[39]

Like other **Kac–Moody algebras**, affine Lie algebras satisfy the **Weyl–Kac character formula**, which expresses the **characters** of the algebra in terms of their **highest weights**.^[40] In the case of affine Lie algebras, the resulting identities are equivalent to the **Macdonald identities**. In particular, for the affine Lie algebra of type $A_1^{(1)}$, associated to the affine symmetric group \tilde{S}_2 , the corresponding Macdonald identity is equivalent to the **Jacobi triple product**.^[41]

Extended affine symmetric group

The affine symmetric group is a subgroup of the *extended affine symmetric group*. The extended group is isomorphic to the wreath product $\mathbb{Z} \wr S_n$. Its elements are *extended affine permutations*: bijections $u: \mathbb{Z} \rightarrow \mathbb{Z}$ that $u(x + n) = u(x) + n$ for all integers x . Unlike the affine symmetric group, the extended affine symmetric group is not a Coxeter group. However, it has a natural generating set that extends the Coxeter generating set for \tilde{S}_n : the *shift operator* τ whose window notation is $\tau = [2, 3, \dots, n, n + 1]$ generates the extended group with the simple reflections, subject to the additional relations $\tau s_i \tau^{-1} = s_{i+1}$.^[7]

Combinatorics of other affine Coxeter groups

The geometric action of the affine symmetric group \tilde{S}_n places it naturally in the family of **affine Coxeter groups**, all of which have a similar geometric action. The combinatorial description of the \tilde{S}_n may also be extended to many of these groups: in **(Eriksson & Eriksson 1998)**, an axiomatic description is given of certain permutation groups acting on \mathbb{Z} (the "George groups", in honor of **George Lusztig**), and it is shown that they are exactly the "classical" Coxeter groups of finite and affine types A, B, C, and D. Thus, the combinatorial interpretations of descents, inversions, etc., carry over in these cases.^[42] Abacus models of minimum-length coset representatives for parabolic quotients have also been extended to this context.^[43]



Acknowledgements

The author thanks the three referees for their many helpful comments, and Max Glick for his help with the less technical summary. The work of the author was supported in part by a Simons Collaboration Grant (634530).

Notes

1. Björner & Brenti (2005), p. 17.
2. Humphreys (1990), p. 17.
3. Humphreys (1990), Chapter 4.
4. Humphreys (1990), Section 4.3.
5. Björner & Brenti (2005), Chapter 8.3.
6. Björner & Brenti (2005), Proposition 8.3.5.
7. Chmutov, Pylyavskyy & Yudovina (2018), Section 1.6.
8. Björner & Brenti (2005), p. 260.
9. Björner & Brenti (2005), p. 15.
10. Humphreys (1990), p. 93.
11. Björner & Brenti (2005), p. 261.
12. Björner & Brenti (2005), p. 208.
13. Björner & Brenti (1996), Cor. 4.7.
14. Björner & Brenti (2005), p. 263.
15. Reiner (1995), p. 2.
16. Petersen (2015), Chapter 14.
17. Reiner (1995), Theorem 6.
18. Lewis et al. (2019), Propositions 1.31 and 4.24.
19. Lewis et al. (2019).
20. Lewis et al. (2019), Theorem 4.25.
21. Lewis et al. (2019), Corollary 2.5.
22. Stembridge (1996), p. 353.
23. Hanusa & Jones (2010), p. 1345.
24. Crites (2010), Theorem 1.
25. Hanusa & Jones (2010), Section 2.2.
26. Lapointe & Morse (2005).
27. Berg, Jones & Vazirani (2009).
28. Björner & Brenti (2005), p. 264.
29. Knutson, Lam & Speyer (2013), Section 2.1.
30. As in (Cameron 1994, Section 3.5).
31. As in, for example, (Beazley et al. 2015), (Lam 2015).
32. Polster (2003), p. 42.
33. Polster (2003), p. 22.
34. Polster (2003), p. 15.
35. Polster (2003), p. 43.
36. Clark & Ehrenborg (2011), Theorem 2.2.
37. Lewis (2020), Section 3.2.
38. Kac (1990), Chapter 3.
39. Kac (1990), Chapter 4.
40. Kac (1990), Chapter 10.
41. Kac (1990), Chapter 12.
42. Björner & Brenti (2005), Chapter 8.
43. Hanusa & Jones (2012).
44. The three positions i, j , and k need not lie in a single window. For example, the affine permutation W in S_4 with window notation $[-4, -1, 1, 14]$ is not fully commutative, because $w(0) = 10, w(3) = 1$, and $w(5) = 0$, even though no four consecutive positions contain a decreasing subsequence of length three.^[23]
45. In a standard Young tableau, entries increase across rows and down columns; in a tabloid, they increase across rows, but there is no column condition.
46. In other words, one might be interested in switching from a left group action to a right action or vice-versa.
47. Not every sequence of n nonnegative integers is a juggling sequence. In particular, a sequence corresponds to a "simple juggling pattern", with one ball caught and thrown at a time, if and only if the function $i \mapsto i + a_i \pmod n$ is a permutation of $\{1, \dots, n\}$.^[33]
48. In some sources, unitary reflections are called *pseudoreflections*.

References

- Beazley, Elizabeth; Nichols, Margaret; Park, Min Hae; Shi, XiaoLin; Youcis, Alexander (2015), "Bijective projections on parabolic quotients of affine Weyl groups", *J. Algeb. Comb.*, **41**: 911–948, doi:10.1007/s10801-014-0559-9
- Berg, Chris; Jones, Brant; Vazirani, Monica (2009), "A bijection on core partitions and a parabolic quotient of the affine symmetric group", *J. Combin. Theory Ser. A*, **116** (8): 1344–1360, doi:10.1016/j.jcta.2009.03.013
- Billiey, Sara C.; Jockusch, William; Stanley, Richard P. (1993), "Some Combinatorial Properties of Schubert Polynomials", *J. Algeb. Comb.*, **2**: 345–374, doi:10.1023/A:1022419800503
- Björner, Anders; Brenti, Francesco (1996), "Affine permutations of type A", *Electron. j. Combin.*, **3** (2): R18, doi:10.37236/1276
- Björner, Anders; Brenti, Francesco (2005), *Combinatorics of Coxeter groups*, Springer, ISBN 978-3540-442387
- Cameron, Peter J. (1994), *Combinatorics: Topics, Techniques, Algorithms*, Cambridge University Press, ISBN 978-0-521-45761-3
- Chmutov, Michael; Pylyavskyy, Pavlo; Yudovina, Elena (2018), "Matrix-ball construction of affine Robinson-Schensted correspondence", *Selecta Math. (N.S.)*, **24** (2): 667–750, doi:10.1007/s00029-018-0402-6
- Clark, Eric; Ehrenborg, Richard (2011), "Excedances of affine permutations", *Advances in Applied Mathematics*, **46**: 175–191, doi:10.1016/j.aam.2009.12.006
- Crites, Andrew (2010), "Enumerating pattern avoidance for affine permutations", *Electron. j. Combin.*, **17** (1): R127, doi:10.37236/399
- Ehrenborg, Richard; Readdy, Margaret (1996), "Juggling and applications to q -analogues", *Discrete Math.*, **157**: 107–125, doi:10.1016/S0012-365X(96)83010-X
- Eriksson, Henrik; Eriksson, Kimmo (1998), "Affine Weyl groups as infinite permutations", *Electron. j. Combin.*, **5**: R18, doi:10.37236/1356
- Green, R.M. (2002), "On 321-Avoiding Permutations in Affine Weyl Groups", *J. Algeb. Comb.*, **15**: 241–252, doi:10.1023/A:1015012524524
- Hanusa, Christopher R.H.; Jones, Brant C. (2010), "The enumeration of fully commutative affine permutations", *Eur. j. Comb.*, **31** (5): 1342–1359, doi:10.1016/j.ejc.2009.11.010
- Hanusa, Christopher R.H.; Jones, Brant C. (2012), "Abacus models for parabolic quotients of affine Weyl groups", *J. Algebra*, **361**: 134–162, doi:10.1016/j.jalgebra.2012.03.029
- Humphreys, James E. (1990), *Reflection groups and Coxeter groups*, Cambridge University Press, ISBN 0-521-37510-X
- Kac, Victor G. (1990), *Infinite dimensional Lie algebras* (3rd ed.), Cambridge University Press, ISBN 0-521-46693-8
- Knutson, Allen; Lam, Thomas; Speyer, David E. (2013), "Positroid varieties: juggling and geometry", *Compositio Math.*, **149**: 1710–1752, doi:10.1112/S0010437X13007240
- Lam, Thomas. (2015), "The shape of a random affine Weyl group element and random core partitions", *Ann. Probab.*, **43** (4): 1643–1662, doi:10.1214/14-AOP915
- Lapointe, Luc; Morse, Jennifer (2005), "Tableaux on $k + 1$ -cores, reduced words for affine permutations, and k -Schur expansions", *J. Combin. Theory Ser. A*, **112** (1): 44–81, doi:10.1016/j.jcta.2005.01.003
- Lewis, Joel Brewster (2020), "A note on the Hurwitz action on reflection factorizations of Coxeter elements in complex reflection groups", *Electron. j. Combin.*, **27** (2): P2.54, doi:10.37236/9351
- Lewis, Joel Brewster; McCammond, Jon; Petersen, T. Kyle; Schwer, Petra (2019), "Computing reflection length in an affine Coxeter group", *Trans. Amer. Math. Soc.*, **371**: 4097–4127, doi:10.1090/tran/7472
- Petersen, T. Kyle (2015), *Eulerian Numbers*, Birkhauser, doi:10.1007/978-1-4939-3091-3, ISBN 978-1-4939-3090-6
- Polster, Burkard (2003), *The Mathematics of Juggling*, Springer, ISBN 0-387-95513-5
- Reiner, Victor (1995), "The distribution of descents and length in a Coxeter group", *Electron. j. Combin.*, **2**: R25, doi:10.37236/1219
- Shephard, G. C.; Todd, J. A. (1954), "Finite unitary reflection groups", *Canad. j. Math.*, **6**: 274–304, doi:10.4153/CJM-1954-028-3
- Shi, Jian-Yi (1986), *Kazhdan-Lusztig cells of certain affine Weyl groups*, Lecture Notes in Mathematics, **1179**, Springer, ISBN 3-540-16439-1
- Shi, Jian-Yi (1991), "The generalized Robinson-Schensted algorithm on the affine Weyl group of type A_{n-1} ", *J. Algebra*, **139** (2): 364–394, doi:10.1016/0021-8693(91)90300-W
- Shi, Jian-Yi (2002), "Certain imprimitive reflection groups and their generic versions", *Trans. Amer. Math. Soc.*, **354** (5): 2115–2129, doi:10.1090/S0002-9947-02-02941-0



- [Stembridge, John](#) (1996), "On the Fully Commutative Elements of Coxeter Groups", *J. Alg. Comb.*, 5: 353–385, doi:10.1007/BF00193185
- [Viennot, G.](#) (1977), "Une forme géométrique de la correspondance de Robinson-Schensted", in [Foata, Dominique](#) (ed.), *Combinatoire et*

représentation du groupe symétrique, Lecture Notes in Mathematics, 579, Springer, pp. 29–58, doi:10.1007/BFb0090011, ISBN 978-3-540-08143-2